Determinants

Recall if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det A = ad - bc$.

Let's interpret this quantity geometrically.

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(v) = Av$.

Let $C = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid 0 \le x, y \le 1 \right\} = \left\{ x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mid 0 \le x, y \le 1 \right\}$ (unit square)

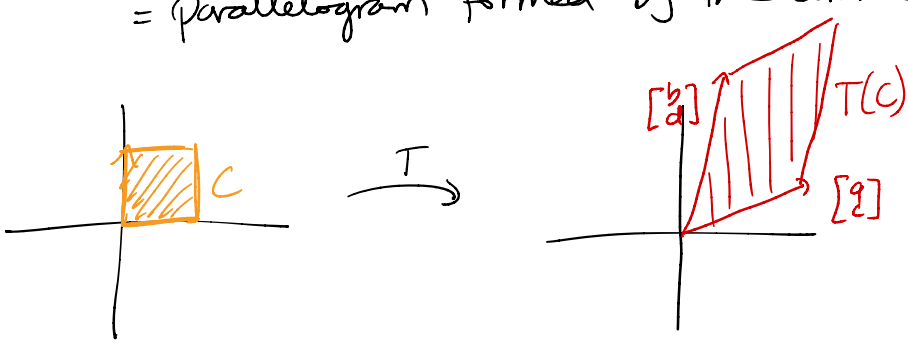
Then $T(C) = \left\{ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \mid 0 \le x, y \le 1 \right\}$

$= \left\{ T\left(x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \mid 0 \le x, y \le 1 \right\}$

$= \left\{ xT\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + yT\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \mid 0 \le x, y \le 1 \right\}$ since T is linear

$= \left\{ x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix} \mid 0 \le x, y \le 1 \right\}$

= parallelogram formed by the columns of A.

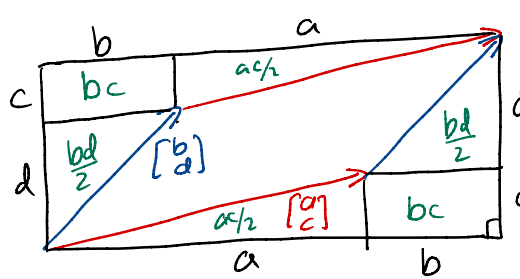


C has area 1. What is the area of $T(C)$?

Assume $a, b, c, d > 0$. Then the area of $T(C)$ is

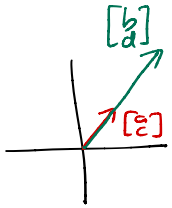
$(a+b)(c+d) - 2bc - ac - bd = ad - bc$

$= \det A$!



More generally, area of $T(C) = |\det A|$.

Important Observation



$\det A = 0 \iff$ parallelogram has area 0

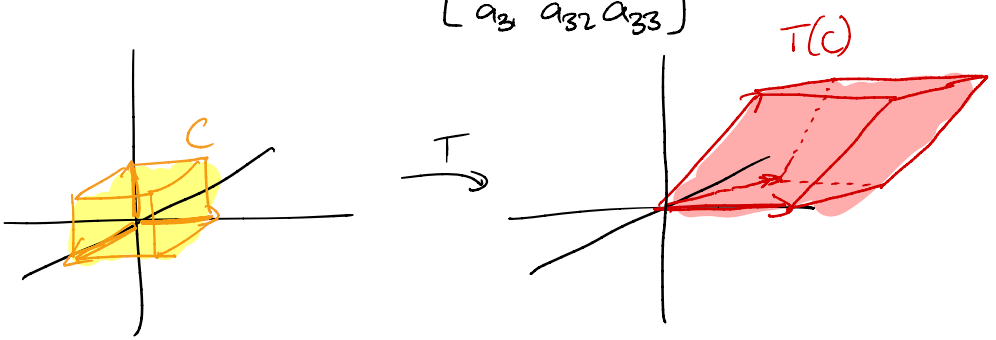
$\iff \begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}$ lie on same line

$\iff \begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}$ are linearly dependent

$\iff$ A is <u>not</u> invertible.

In general, if A is $n \times n$ and $T: \mathbb{R}^n \to \mathbb{R}^n$ is its associated linear transformation, then $|\det A|$ = volume of $T(C)$, where C is the "unit cube"

Ex: For 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$



$$\det A = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

(triple product from multivariable calculus)

More generally, if R is a region in $\mathbb{R}^n$, then Volume$(T(R))$ = $|\det A|$ Volume(R). So:

$\det A$ measures how much T changes "volume"

Determinant Axioms:

1. $\det I = 1$
2. Row swaps change sign of det

Ex: $\det\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} = -\det\begin{bmatrix}7&8&9\\4&5&6\\1&2&3\end{bmatrix}$ (Swap R1 & R3)

3. The determinant is a linear transformation of each row:

- $\det\begin{bmatrix}\vdots\\a^T+b^T\\\vdots\end{bmatrix} = \det\begin{bmatrix}\vdots\\a^T\\\vdots\end{bmatrix} + \det\begin{bmatrix}\vdots\\b^T\\\vdots\end{bmatrix}$
- $\det\begin{bmatrix}\vdots\\ca^T\\\vdots\end{bmatrix} = c\det\begin{bmatrix}\vdots\\a^T\\\vdots\end{bmatrix}$

Ex: $\det\begin{bmatrix}1&2&3\\0&1&2\\1&1&1\end{bmatrix} + \det\begin{bmatrix}1&2&3\\2&1&-1\\1&1&1\end{bmatrix} = \det\begin{bmatrix}1&2&3\\2&2&1\\1&1&1\end{bmatrix}$

$\det\begin{bmatrix}2&4&5\\3&6&9\\0&1&2\end{bmatrix} = 3\det\begin{bmatrix}2&4&5\\1&2&3\\0&1&2\end{bmatrix}$

Check: These axioms hold for 2x2 determinants
(Recitation + HW)

From these axioms, we will see that we can determine the determinant of any matrix

Proposition 1: If $D = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & d_n \end{bmatrix}$, then $\det D = d_1 d_2 \cdots d_n$

(diagonal matrix)

Proof:

$$\det D = d_1 \det \begin{bmatrix} 1 & & 0 \\ & d_2 & \\ 0 & & d_n \end{bmatrix} \quad \text{(Axiom 3)}$$

$$\vdots$$

$$= d_1 d_2 \cdots d_n \det \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \quad \text{(Axiom 3)}$$

$$= d_1 d_2 \cdots d_n. \quad \text{(Axiom 1)}$$

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Proposition 2: If A has two identical rows, then $\det A = 0$

Proof:

Let $A = \begin{bmatrix} \vdots \\ a^T \\ \vdots \\ a^T \\ \vdots \end{bmatrix}$.

Then $\det A = \det \begin{bmatrix} \vdots \\ a^T \\ \vdots \\ a^T \\ \vdots \end{bmatrix}$

$$= -\det \begin{bmatrix} \vdots \\ a^T \\ \vdots \\ a^T \\ \vdots \end{bmatrix} \quad \text{(Axiom 2)}$$

$$= -\det A$$

$$\Rightarrow 2 \det A = 0 \Rightarrow \det A = 0$$

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Proposition 3: Row replacement / Elimination does not change det.

Proof:

$$\det\begin{bmatrix} \vdots \\ a^T \\ \vdots \\ b^T + ca^T \\ \vdots \end{bmatrix} = \det\begin{bmatrix} \vdots \\ a^T \\ \vdots \\ b^T \\ \vdots \end{bmatrix} + c\det\begin{bmatrix} \vdots \\ a^T \\ \vdots \\ a^T \\ \vdots \end{bmatrix} \quad \text{(Axiom 3)}$$

$$= \det\begin{bmatrix} \vdots \\ a^T \\ \vdots \\ b^T \\ \vdots \end{bmatrix} \quad \text{(Prop 2)}$$

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Proposition 4: If A has a row of zeros, then $\det A = 0$

Proof:

$$\det\begin{bmatrix} a^T \\ \vdots \\ 0 \\ \vdots \end{bmatrix} \overset{\text{Prop 3}}{=} \det\begin{bmatrix} a^T \\ \vdots \\ a^T \\ \vdots \end{bmatrix} \overset{\text{Prop 2}}{=} 0$$

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Proposition 5: If A is triangular, then $\det A$ is the product of the diagonal entries.

Proof: We will prove for upper triangular. Lower triangular is similar.

Case 1: If diagonal entries all nonzero

$$\det\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{12} & & \vdots \\ 0 & & \ddots & \\ & & & a_{1n} \end{bmatrix} \overset{\text{Prop 3}}{=} \det\begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{1n} \end{bmatrix} \overset{\text{Prop 1}}{=} a_{11} \cdots a_{1n}$$

Case 2: $a_{ii} = 0$ for some i.

$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \ddots & & \vdots \\ 0 & & 0 & \\ & & & a_{nn} \end{bmatrix} \overset{\text{use elimination}}{=} \det \begin{bmatrix} a_{11} & & & 0 \\ & \ddots & & \\ 0 & \cdots & 0 & \cdots 0 \\ & & & a_{nn} \end{bmatrix} \quad \text{(Prop 3)}$$

lowest 0 on diagonal

zero row

$$= 0 \quad \text{(prop 4)}$$

$$= a_{11} \cdots a_{nn} \quad \text{since } a_{ii} = 0$$

Ex: $\det \begin{bmatrix} -1 & 15 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 6 \end{bmatrix} = (-1)(2)(6) = -12$

We can now find the determinant of any matrix:

Reduce A to U.T. form U via elimination & row swaps

Then: $\det A = (-1)^{\#\text{Swaps}} \det(U) = (-1)^{\#\text{Swaps}} u_{11} \cdots u_{nn}$

$\det A$: Axiom 2 & Prop 3; $\det(U)$: Prop 5

Ex: $\det \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ 1 & 3 & 4 \end{bmatrix} = \det \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 4 & 2 \end{bmatrix}$ (Row Replacement)

$$= -\det \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(Row Swap)}$$

$$= -(1)(4)(1) = -4$$